Acyclic and Star Colorings of Joins of Graphs and an Algorithm for Cographs

Andrew Lyons

Computation Institute, University of Chicago and
Mathematics and Computer Science Division, Argonne National Laboratory

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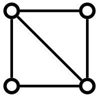
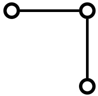
Subgraphs and Induced Subgraphs

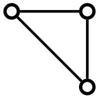
$$G = \{V, E\}$$





Subgraph

$G' = \{V', E'\}$ where $V' \subseteq V$ and $E' \subseteq E$



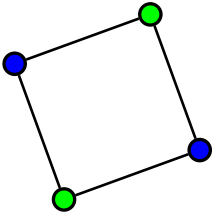
Induced Subgraph

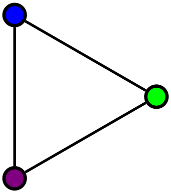
$G' = \{V', E'\}$ where $V' \subseteq V$ and E' consists of all edges with both endpoints in V'
(vertex-induced subgraph)

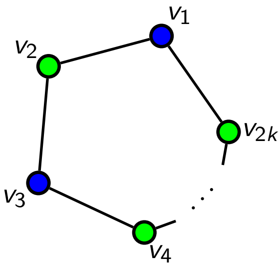
Outline

Coloring

proper vertex coloring



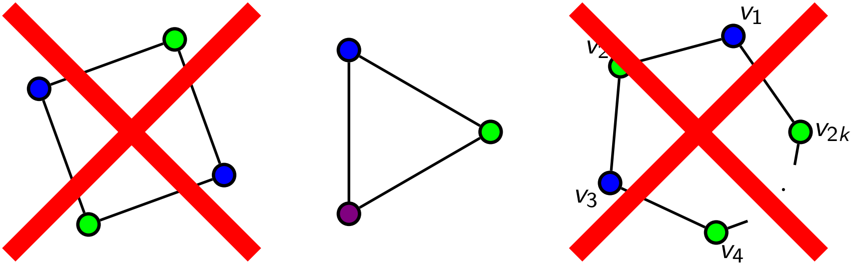




chromatic number $\chi(G)$

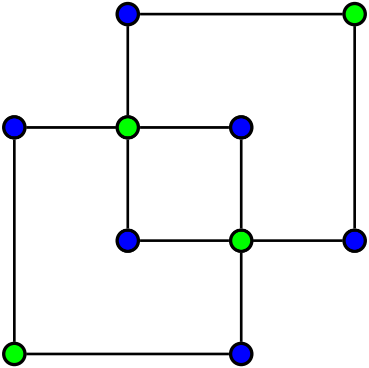
Acyclic Coloring

proper vertex coloring without bichromatic cycles



acyclic chromatic number $\chi_a(G) \geq \chi(G)$

Acyclic Coloring – No Bichromatic Cycles



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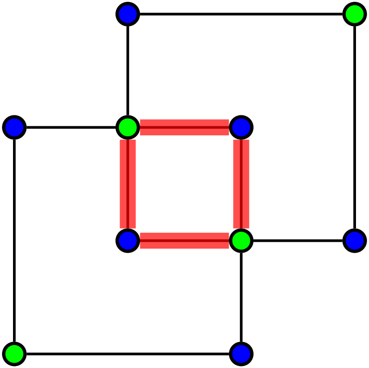

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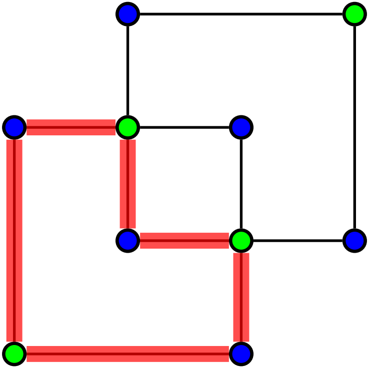

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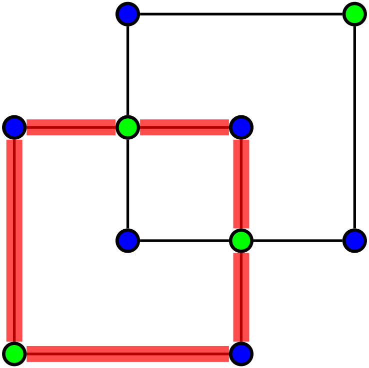

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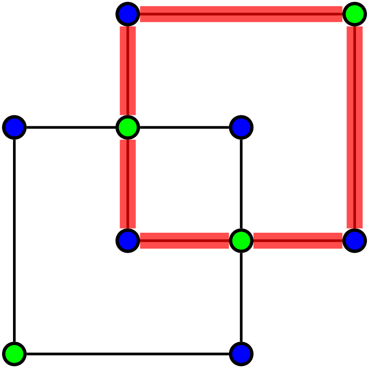

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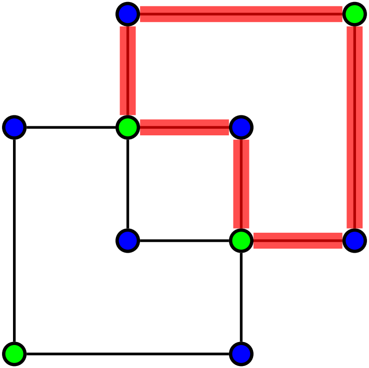

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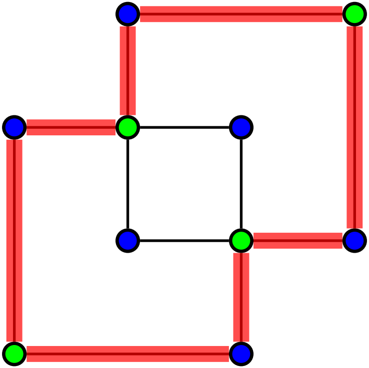

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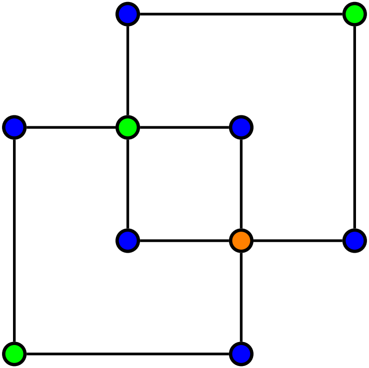

$\chi_a(G) = 3$

Acyclic Coloring – Definitions

A proper vertex coloring such that . . .

Original Definition

. . . every (even) cycle uses ≥ 3 colors.

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A proper vertex coloring such that . . .

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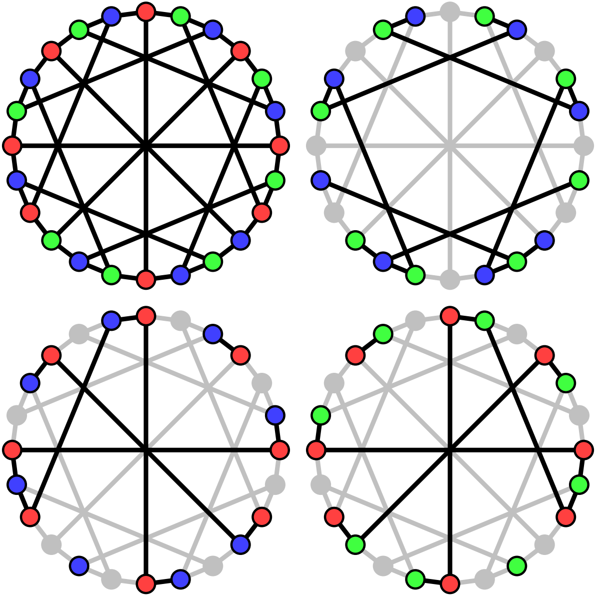
Bichromatic Induced Subgraphs

. . . the subgraph induced by any two color classes is a disjoint collection of trees (a *forest*).



credit: Claudio Rocchini (GNU Free Documentation License)

http://commons.wikimedia.org/wiki/File:Acyclic_coloring.svg

Acyclic Coloring – Algorithms

Chordal Graphs

Solvable in linear time for this class of graphs.
(In fact, every coloring of a chordal graph is also an acyclic coloring.)
(Gebremedhin, Pothen, Tarafdar, & Walther 2009).

Acyclic Coloring – Algorithms

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Bounded maximum degree $\Delta(G)$

- If $\Delta(G) \leq 3$, then G can be acyclically colored using 4 colors or fewer in linear time (Skulrattanakulchai 2004).
- If $\Delta(G) \leq 5$, then G can be acyclically colored using 9 colors or fewer in linear time (Fertin & Raspaud 2008).

Acyclic Coloring – Complexity

NP-Complete to determine whether $\chi_a(G) \leq 3$
(Kostochka 1978)

NP-hard even when restricted to bipartite graphs
(Coleman & Cai 1986)

Coloring

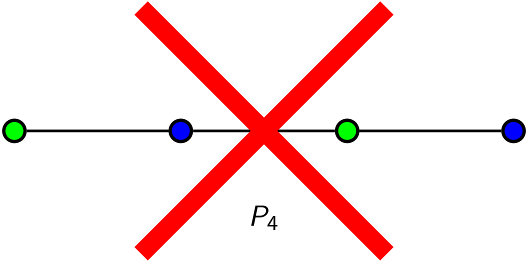
proper vertex coloring

P_4

chromatic number $\chi(G)$

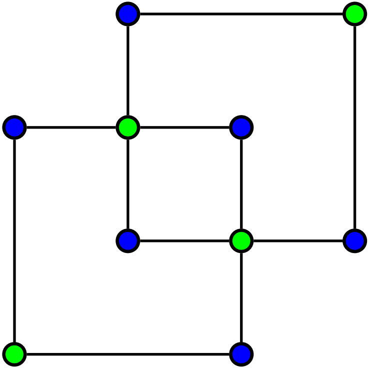
Star Coloring

proper vertex coloring with no bichromatic P_4
(That's *every* P_4, not just the induced ones)

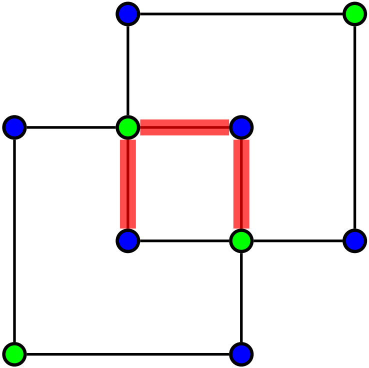


star chromatic number $\chi_s(G) \geq \chi_a(G) \geq \chi(G)$
(A bichromatic cycle implies a bichromatic P_4)

Star Coloring



Star Coloring



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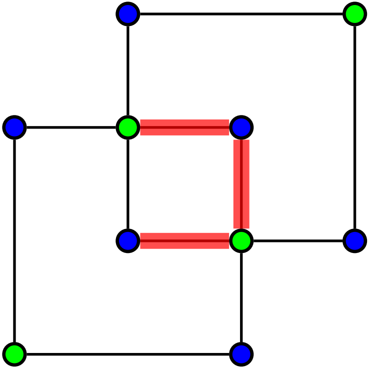

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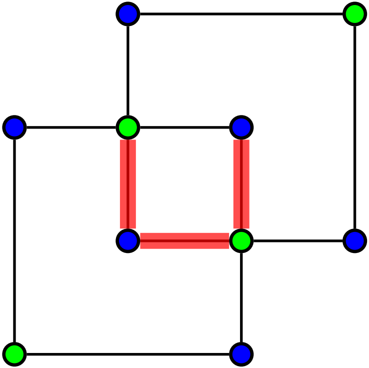

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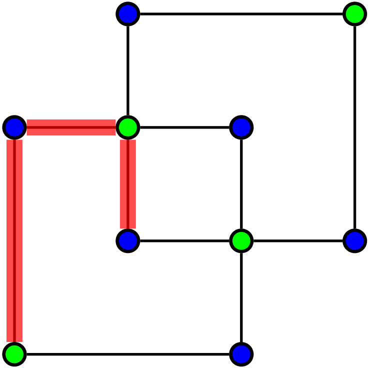
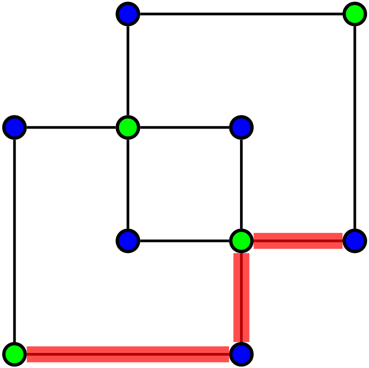
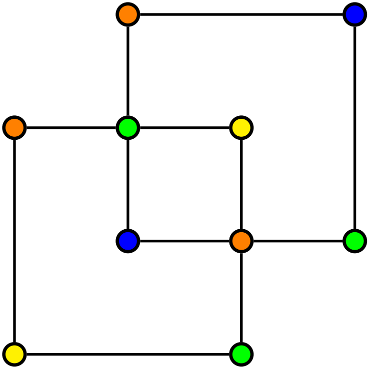


Star Coloring



Star Coloring



$\chi_s(G) = 4$ (I think)

Star Coloring – Definitions

A proper vertex coloring such that . . .

Original Definition

. . . every P_4 uses ≥ 3 colors.

Star Coloring – Definitions

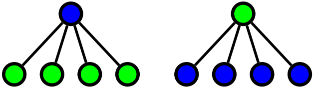
A proper vertex coloring such that ...

Original Definition

... every P_4 uses ≥ 3 colors.

Bichromatic Induced Subgraphs

... the subgraph induced by any two color classes is a disjoint collection of *stars*.



Star Coloring – Complexity

NP-Complete to determine whether $\chi_s(G) \leq 3$ for planar bipartite graphs
(Albertson, Chappell, Kierstead, Kündgen, & Ramamurthi 2004)

NP-hard when restricted to bipartite graphs
(Coleman & Moré 1984)

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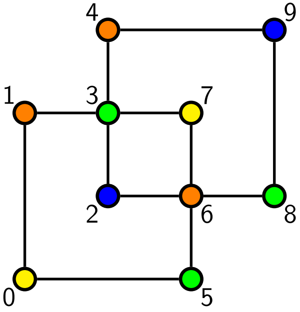
Open Problem

For a split graph $G, \chi_s(G)$ is either $\omega(G)$ or $\omega(G) + 1$.
What is the complexity of determining this?

Outline

Star Coloring – Direct Hessian Computation

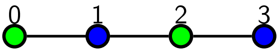
$$
\begin{bmatrix}
h_{00} & h_{01} & & & & h_{05} & & & & \\
h_{10} & h_{11} & & h_{13} & & & & & & \\
& & h_{22} & h_{23} & & & h_{26} & & & \\
& h_{31} & h_{32} & h_{33} & h_{34} & & & h_{37} & & \\
& & & h_{43} & h_{44} & & & & & h_{49} \\
h_{50} & & & & & h_{55} & h_{56} & & & \\
& & h_{62} & & & h_{65} & h_{66} & h_{67} & h_{68} & \\
& & & h_{73} & & & h_{76} & h_{77} & & \\
& & & & & & h_{86} & & h_{88} & h_{89} \\
& & & & h_{94} & & & & h_{98} & h_{99}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
$$



$$
\begin{bmatrix}
 & h_{01} & h_{05} & h_{00} \\
 & h_{11} & h_{13} & h_{10} \\
h_{22} & h_{26} & h_{23} & \\
h_{32} & h_{31} + h_{34} & h_{33} & h_{37} \\
h_{49} & h_{44} & h_{43} & \\
 & h_{56} & h_{55} & h_{50} \\
h_{62} & h_{66} & h_{65} + h_{68} & h_{67} \\
 & h_{76} & h_{73} & h_{77} \\
h_{89} & h_{86} & h_{88} & \\
h_{99} & h_{94} & h_{98} &
\end{bmatrix}
$$

Star Coloring – Direct Hessian Computation

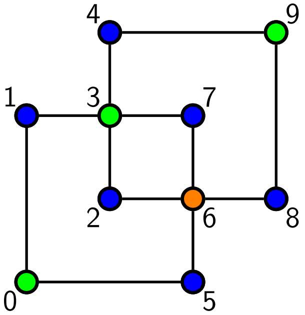
$$\begin{bmatrix} h_{00} & h_{01} & & \\ h_{10} & h_{11} & h_{12} & \\ & h_{21} & h_{22} & h_{23} \\ & & h_{32} & h_{33} \end{bmatrix}$$



$$\begin{bmatrix} h_{00} & h_{01} \\ h_{10} + h_{12} & h_{11} \\ h_{22} & h_{21} + h_{23} \\ h_{32} & h_{33} \end{bmatrix}$$

Acyclic Coloring – Indirect Hessian Computation

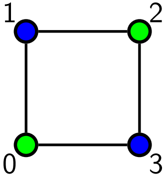
$$
\begin{bmatrix}
h_{00} & h_{01} & & & & h_{05} & & & & \\
h_{10} & h_{11} & & h_{13} & & & & & & \\
& & h_{22} & h_{23} & & & h_{26} & & & \\
& h_{31} & h_{32} & h_{33} & h_{34} & & & h_{37} & & \\
& & & h_{43} & h_{44} & & & & & h_{49} \\
h_{50} & & & & & h_{55} & h_{56} & & & \\
& & h_{62} & & & h_{65} & h_{66} & h_{67} & h_{68} & \\
& & & h_{73} & & & h_{76} & h_{77} & & \\
& & & & & & h_{86} & & h_{88} & h_{89} \\
& & & & h_{94} & & & & h_{98} & h_{99}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$



$$
\begin{bmatrix}
h_{01} + h_{05} & & h_{00} \\
h_{11} & & h_{10} + h_{13} \\
h_{22} & h_{26} & h_{23} \\
h_{31} + h_{32} + h_{34} + h_{37} & & h_{33} \\
h_{44} & & h_{43} + h_{49} \\
h_{55} & h_{56} & h_{50} \\
h_{62} + h_{65} + h_{67} + h_{68} & h_{66} & \\
h_{77} & h_{76} & h_{73} \\
h_{88} & h_{86} & h_{89} \\
h_{94} + h_{98} & & h_{99}
\end{bmatrix}
$$

Acyclic Coloring – Indirect Hessian Computation

$$\begin{bmatrix} h_{00} & h_{01} & & h_{03} \\ h_{10} & h_{11} & h_{21} & \\ & h_{12} & h_{22} & h_{23} \\ h_{30} & & h_{32} & h_{33} \end{bmatrix}$$



$$\begin{bmatrix} h_{01} + h_{03} & h_{00} \\ h_{11} & h_{10} + h_{21} \\ h_{12} + h_{23} & h_{22} \\ h_{33} & h_{30} + h_{32} \end{bmatrix}$$

Coloring for Efficient Derivative Matrix Computation

Hessian Computation

Star Coloring: Direct computation

Acyclic coloring: Indirect (substitution) computation

Jacobian Computation

Distance-2 Coloring: Direct, 1-dimensional computation

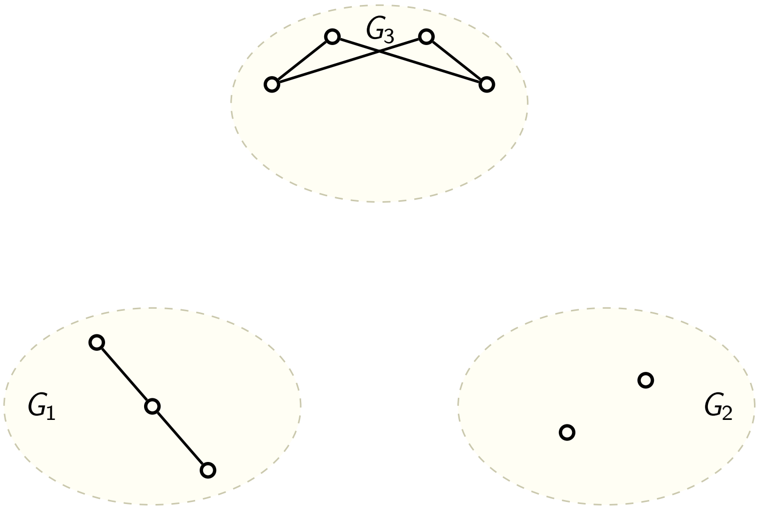
Star Bicoloring: Direct, 2-dimensional computation

Acyclic Bicoloring: Indirect (substitution), 2-dimensional computation

A. Gebremedhin, F. Manne, A. Pothen, What Color Is Your Jacobian? Graph Coloring for Computing Derivatives, *SIAM Review* **47**:4 (2005).

Outline

The Join Operation $*$



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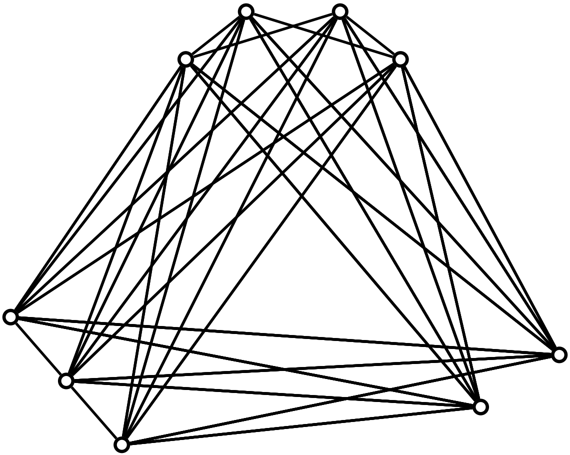

$$\bigotimes_{i=1}^{3} G_i$$

The Main Theorem

Theorem

Let $\{G_i = (V_i, E_i)\}_{i\in\mathcal{I}}$ *be a finite collection of graphs. Then*

(i) $$\chi_a\left(\bigotimes_{i\in\mathcal{I}} G_i\right) = \sum_{i\in\mathcal{I}} \chi_a(G_i) + \min_{j\in\mathcal{I}}\left\{\sum_{i\in\mathcal{I}, i\neq j} (|V_i| - \chi_a(G_i))\right\};$$

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(i) $$\chi_a\left(\bigotimes_{i\in\mathcal{I}} G_i\right) = \sum_{i\in\mathcal{I}} \chi_a(G_i) + \min_{j\in\mathcal{I}}\left\{\sum_{i\in\mathcal{I}, i\neq j} (|V_i| - \chi_a(G_i))\right\};$$

(ii) $$\chi_s\left(\bigotimes_{i\in\mathcal{I}} G_i\right) = \sum_{i\in\mathcal{I}} \chi_s(G_i) + \min_{j\in\mathcal{I}}\left\{\sum_{i\in\mathcal{I}, i\neq j} (|V_i| - \chi_s(G_i))\right\}.$$

The Binary Case

$$(G_1 * G_2) * G_3 = G_1 * (G_2 * G_3) = (G_1 * G_3) * G_2 = \cdots$$

The join operation is commutative and associative
$\Rightarrow$ we will work with the binary case.

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The join operation is commutative and associative
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Lemma

Let $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *be graphs. Then*

(i)
$$\begin{aligned} \chi_a(G_1 * G_2) &= \chi_a(G_1) + \chi_a(G_2) \\ &\quad + \min\{|V_1| - \chi_a(G_1), |V_2| - \chi_a(G_2)\}\,; \end{aligned}$$

The Binary Case

$$(G_1 * G_2) * G_3 = G_1 * (G_2 * G_3) = (G_1 * G_3) * G_2 = \cdots$$

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Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. Then

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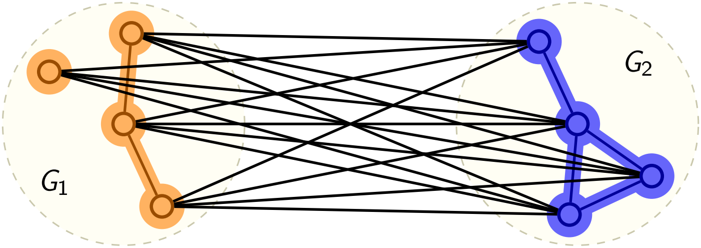
Proof of Lemma

- G_1 and G_2 are induced subgraphs of $G_1 * G_2$.
- G_1 and G_2 cannot share any colors.

$$\chi_a(G_1 * G_2) \geq \chi_a(G_1) + \chi_a(G_2)$$



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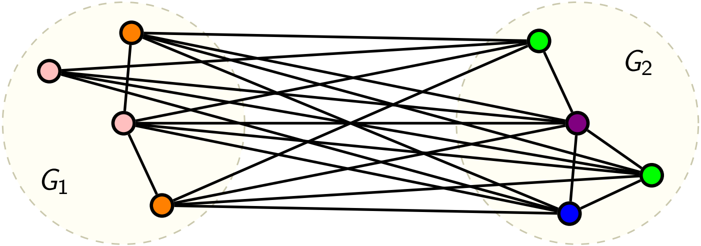
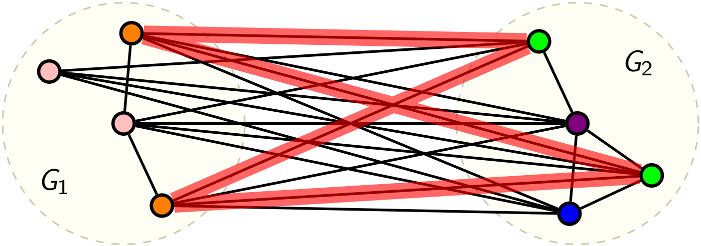


$\chi_a(G_1) = 2$ $\chi_a(G_2) = 3$

Proof of Lemma

$$\chi_a(G_1 * G_2) \gneq \chi_a(G_1) + \chi_a(G_2)$$



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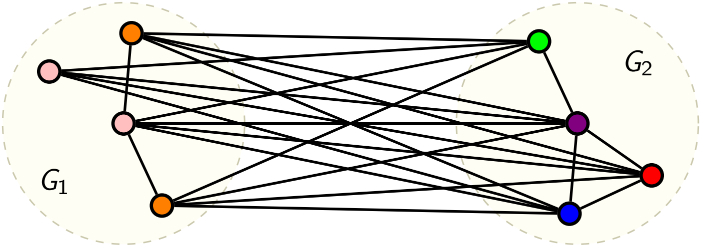

$\chi_a(G_1) = 2,\ |V_1| = 4$ $\chi_a(G_2) = 3,\ |V_2| = 4$

Proof of Lemma

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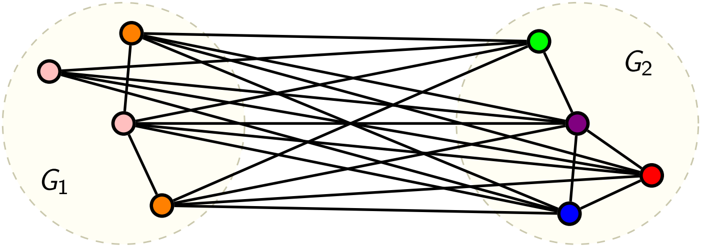
$$\chi_s(G_1 * G_2) = \chi_s(G_1) + \chi_s(G_2) + \min\{|V_1| - \chi_s(G_1), |V_2| - \chi_s(G_2)\}$$



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Outline

Cographs

Forbidden subgraph characterization

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Restricted Coloring Characterization

A graph is a cograph if and only if every acyclic coloring is also a star coloring.

Cographs

Recursive Definition

A graph G is a cograph if and only if one of the following is true.

(i) $|V| = 1$;

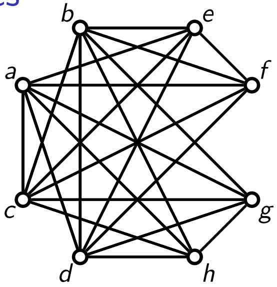
(ii) there exists a collection $\{G_i\}_{i \in \mathcal{I}}$ of cographs such that

$$G = \bigcup_{i \in \mathcal{I}} G_i \quad \text{(disjoint union)};$$

(iii) there exists a collection $\{G_i\}_{i \in \mathcal{I}}$ of cographs such that

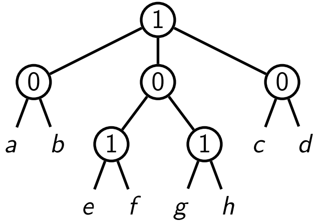
$$G = \bigotimes_{i \in \mathcal{I}} G_i \quad \text{(join)}.$$

Cographs and Cotrees



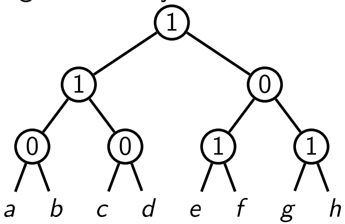
Canonical cotree

Unique (up to isomorphism)



Binary cotree

Algorithmically convenient

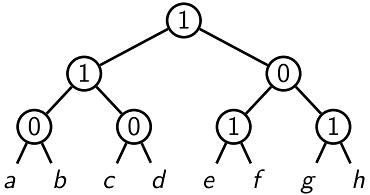


Acyclic and Star Coloring Cographs

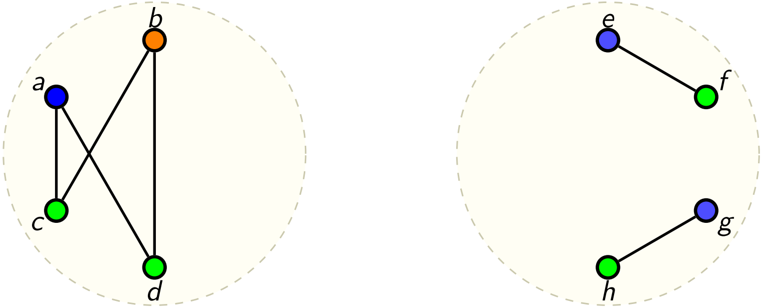
Theorem

An optimal acyclic coloring of a cograph can be found in linear time. Furthermore, the obtained coloring is also an optimal star coloring.

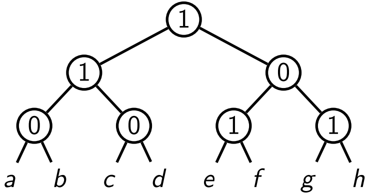
Example



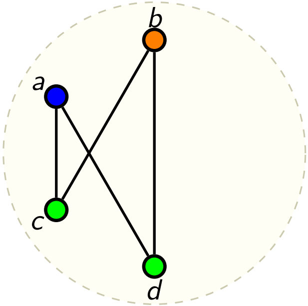
$$\chi_a(G_1 * G_2) \quad = \quad \chi_a(G_1) + \chi_a(G_2) + \min\{|V_1| - \chi_a(G_1), |V_2| - \chi_a(G_2)\}$$



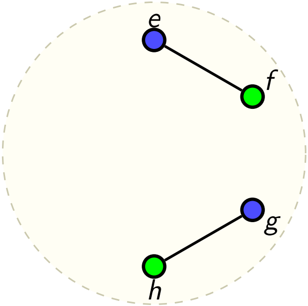
Example



$$\chi_a(G_1 * G_2) \quad = \quad \chi_a(G_1) + \chi_a(G_2) + \min\{|V_1| - \chi_a(G_1), |V_2| - \chi_a(G_2)\}$$

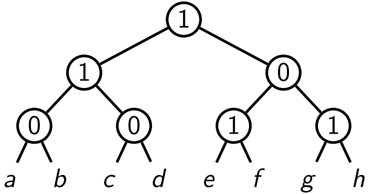


$|V_1| = 4,\ \chi_a(G_1) = 3$

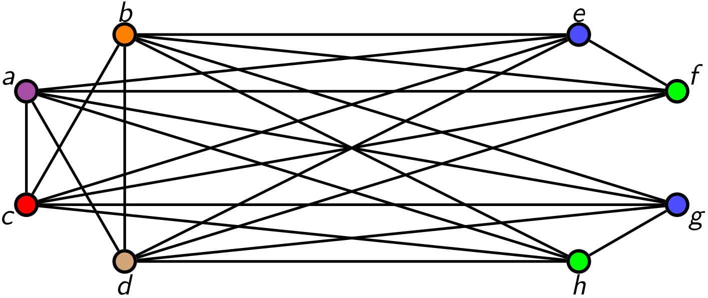


$|V_2| = 4,\ \chi_a(G_1) = 2$

Example



$$\begin{aligned}\chi_a(G_1 * G_2) &= \chi_a(G_1) + \chi_a(G_2) + \min\{|V_1| - \chi_a(G_1), |V_2| - \chi_a(G_2)\} \\ &= 3 + 2 + \min\{4-3, 4-2\} \\ &= 6\end{aligned}$$



$|V_1| = 4,\ \chi_a(G_1) = 3$

$|V_2| = 4,\ \chi_a(G_1) = 2$

Outline

Future Work

Extension to other graph classes

Tree-cographs Same operations as cographs, but start with trees rather than single isolated vertices

P_4-sparse No set of five vertices induces more than one P_4. (Generalize by adding a third composition operation.)

P_4-lite ...

P_4-extendible ...

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Tree-cographs Same operations as cographs, but start with trees rather than single isolated vertices

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Other Decompositions

Modular

Split

Clique

Tree

...

Thank You!

Questions?